Available online at www.sciencedirect.com
ScienceDirect
International Journal of Heat and Fluid Flow 27 (2006) 867–877
International Journal of HEAT AND FLUID FLOW
www.elsevier.com/locate/ijhff

Stochastic modelling of conjugate heat transfer in near-wall turbulence

Jacek Pozorski [a,*], Jean-Pierre Minier [b]

[a] Institute of Fluid-Flow Machinery, Polish Academy of Sciences, Fiszera 14, 80952 Gdańsk, Poland
[b] Research and Development Division, Electricité de France, 6 quai Watier, 78400 Chatou, France

Received 10 October 2005; received in revised form 31 January 2006; accepted 4 March 2006
Available online 14 June 2006

Abstract

The paper addresses the conjugate heat transfer in turbulent flows with temperature assumed to be a passive scalar. The Lagrangian approach is applied and the heat transfer is modelled with the use of stochastic particles. The intensity of thermal fluctuations in near-wall turbulence is determined from the scalar probability density function (PDF) with externally provided dynamical statistics. A stochastic model for the temperature field in the wall material is proposed and boundary conditions for stochastic particles at the solid–fluid interface are formulated. The heated channel flow with finite-thickness walls is considered as a validation case. Computation results for the mean temperature profiles and the variance of thermal fluctuations are presented and compared with available DNS data.

Keywords: Wall turbulence; Conjugate heat transfer; Thermal fluctuations; Stochastic modelling

1. Introduction

The issue of heat transfer in turbulent flows is of utmost practical importance for many engineering problems. Yet, physically sound turbulence modelling at an acceptable computational cost is inherently difficult (Launder, 1996; Nagano, 2002). Detailed physics includes a complex evolution of the temperature field due to the near-wall vortical flow structures (Robinson, 1991) but the statistical approach is still very useful in practical computations. As discussed in the literature (Sommer et al., 1994; Kong et al., 2000; Tiselj et al., 2001a,b), the type of boundary condition for temperature (isothermal wall, isoflux wall, conjugate heat transfer) directly influences the intensity of thermal fluctuations at the fluid–solid interface. Physically, the intensity depends also on the molecular Prandtl number and on the wall material properties. At the level of one-point, one-time statistical averages, the temperature variance and the turbulent heat flux are of direct engineering interest. Another important statistic is the frequency spectrum of near-wall thermal fluctuations. A motivation to undertake the present work originated from some industrial situations of conjugate heat transfer where the mechanical integrity of certain solid structures can be impaired by large and/or rapid temperature changes induced by neighbouring fluid. Therefore, to achieve optimal design and to avoid thermal fatigue of wall material, a fairly detailed estimation of dynamical and thermal statistics of turbulence in the near-wall region is desirable.

In several DNS studies of wall-bounded flows with heat transfer (Kasagi et al., 1992; Kawamura et al., 1998, 1999; Na et al., 1999), the isothermal b.c. (constant wall temperature) were imposed, resulting in zero level of thermal fluctuations there. On the other hand, studies with the isoflux (constant heat flux) b.c. to determine the non-zero level of temperature fluctuations in the boundary layer and the solid wall are rare both in experiments and in RANS computations. Yet, in a couple of subsequent papers on the DNS with heat transfer (Lu and Hetsroni, 1995; Kong et al., 2000; Tiselj et al., 2001a), the isoflux b.c. was imposed as a constant gradient of the instantaneous temperature at every computational node located at the wall, leading to a non-zero wall level of thermal fluctuations.

* Corresponding author. Fax: +48 58 3416144.
E-mail address: jp@imp.gda.pl (J. Pozorski).

0142-727X/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.ijheatfluidflow.2006.03.012

The isoflux condition constitutes one limit case of a more general conjugate fluid–solid heat transfer, namely that of a thin and/or poorly conducting wall, with a thick and/or well-conducting wall making another limit case, that of isothermal boundary. As shown by Tiselj et al. (2001b), the profile of fluctuating temperature variance in a fluid for the general case lies between the two limit profiles (isoflux and isothermal).

The conjugate heat transfer problem has to be formulated in a coupled way, with the flow solution and the temperature field on the fluid side coupled to an unsteady heat conduction in the solid. The experimental and numerical studies of this general case are scarcely reported in the literature. Mosyak et al. (2001) carried out an experimental study of the wall temperature fluctuations under different boundary conditions; they also examined coherent structures of the temperature field (thermal streaks). Tiselj et al. (2001b) performed a DNS of the turbulent heat transfer with unsteady heat conduction in the solid; they reported a number of results, including the dependence of the r.m.s. temperature fluctuations on several relevant parameters.

Concerning theoretical studies, Polyakov (1974) solved a simplified problem for temperature fluctuations in the viscous sublayer of turbulent flow next to a heat-conductive wall; therefrom, he derived analytical expressions for the near-wall frequency spectrum of temperature and predicted the wall level of temperature fluctuations, θ_{rms}. Albeit usually obtained with simplifications, analytical expressions are always most welcome for a better understanding, including dependence on relevant parameters. For example, Polyakov made an important observation concerning the dominant frequencies in the temperature spectrum. The inverses of those frequencies (characteristic time scales) were found to fall roughly in the interval of 100–300 in wall units.

To the best of the authors' knowledge, statistical RANS modelling attempts for turbulent flows aiming at resolving the intensity of thermal fluctuations in the boundary layer with the wall b.c. different than isothermal are quite rare and, apparently, none determined the temperature fluctuations in the wall material itself. Sommer et al. (1994) studied the sensitivity of near-wall thermal fluctuations in a fluid to the type of boundary; yet, they took the prediction of Polyakov (1974) as the wall b.c. for the $\langle\theta^2\rangle$ equation in the iso-flux case. An interesting model with two-dimensional streaky structures and unsteady heat conduction in the wall was proposed by Kasagi et al. (1989). They also used as reference the parametric relationship of Polyakov.

In the present paper, the conjugate heat transfer problem is formulated in the stochastic Lagrangian approach. One building block of the modelling scheme proposed here is the scalar probability density function (PDF) method (cf. Pope, 2000) with down to the wall integration (Pozorski et al., 2004). It is applied in fluid with the use of externally provided turbulence statistics; the temperature is assumed to be a dynamically passive scalar. Another building block are model developments to estimate temperature fluctuations within the wall material. Moreover, the particle treatment of the material interface is put forward. The reason for exploring the Lagrangian stochastic approach is twofold. First, it allows us to obtain quantitative results for the temperature statistics both in fluid and in solid material; as a spin-off, closed Eulerian equations for the temperature variance are obtained. Second, the methodology developed here is readily extended to the case of structural turbulence models with instantaneous flow structures resolved (cf. conclusion). Another aim of the paper is to validate the model in the case of the fully developed heated channel flow. We compare the PDF computation results to some reference DNS data, discuss the sensitivity of near-wall thermal fluctuations to the boundary conditions, the molecular Prandtl number, and the wall material properties.

2. Governing equations

Under suitable simplifying assumptions, the internal energy balance in the flow reduces to the diffusion–advection equation obeyed by instantaneous fluid temperature T as an additional, passive scalar variable:

$$\frac{\partial T}{\partial t} + U_j \frac{\partial T}{\partial x_j} = \alpha_1 \frac{\partial^2 T}{\partial x_j \partial x_j}. \tag{1}$$

The instantaneous temperature in the solid (the wall material) obeys the unsteady heat equation:

$$\frac{\partial T}{\partial t} = \alpha_2 \frac{\partial^2 T}{\partial x_j \partial x_j}. \tag{2}$$

The subscripts 1 and 2 refer to fluid and solid, respectively; in particular, α_1 and α_2 stand for the respective thermal diffusivity coefficients, $\alpha = \lambda/(\rho c_p)$; moreover, λ_1 and λ_2 are respective heat conductivity coefficients in fluid and solid.

The geometric configuration considered in the paper is a flat fluid–solid interface, i.e., the plane $y = 0$, with solid material located in the region $y < 0$. Consistency conditions at the interface are the continuity of the instantaneous temperature and of the heat flux:

$$T|_{y=0^+} = T|_{y=0^-} \quad \text{and} \quad \lambda_1 \frac{\partial T}{\partial y}|_{y=0^+} = \lambda_2 \frac{\partial T}{\partial y}|_{y=0^-}. \tag{3}$$

For subsequent use, the so-called thermal activity ratio is defined as

$$K = \sqrt{\frac{(\rho c_p \lambda)_1}{(\rho c_p \lambda)_2}} = \frac{\lambda_1}{\lambda_2}\sqrt{\frac{\alpha_2}{\alpha_1}}. \tag{4}$$

The mean and fluctuating temperatures in fluid and solid are introduced through $T = \langle T\rangle + \theta$. In the statistical description of turbulent fluid motion, details of the (otherwise deterministic) relevant fields are skipped and only averaged variables are preserved. Notwithstanding the zero velocity field, the same concept of averaging is applied to the temperature in a solid and proves useful upon a clo-

ser examination. There, the stochastic character of the temperature field originates from a random forcing exerted by the fluid on the separating interface. The meaning of ensemble averages $\langle\cdot\rangle$ in a solid is possibly best explained as the time average, like in the streaky-structure model of Kasagi et al. (1989). Concerning the notion of fluctuations in a fluid, the averaging over realisations is best understood in the PDF approach; the time average or averaging over directions of homogeneity are mainly used in DNS/LES, while only statistics of fluctuations make sense as variables in RANS.

Although the moment (or mean) equations are not solved in the stochastic approach, they will be recalled here for the sake of completeness. Due to linearity of Eq. (2), the mean temperature equation in the solid requires no specific closure. However, this is not the case of higher-order moments. The exact equation for the fluctuating temperature variance $\langle\theta^2\rangle$ is derived from Eq. (2) in the same way as done for turbulent fluid flow; yet, in a solid it has a simpler form:

$$\frac{\partial\langle\theta^2\rangle}{\partial t}=\alpha_2\frac{\partial^2\langle\theta^2\rangle}{\partial x_j\partial x_j}-2\alpha_2\left\langle\frac{\partial\theta}{\partial x_j}\frac{\partial\theta}{\partial x_j}\right\rangle. \tag{5}$$

The first RHS term of (5) represents the molecular diffusion whereas the second (unclosed) term stands for the destruction rate of $\langle\theta^2\rangle$.

3. Stochastic modelling of thermal fluctuations in solid

3.1. Problem formulation; idea of the model

Before a stochastic approach to the conjugate heat transfer is detailed, we consider first the issue of temperature fluctuations within a layer of solid material only. The fluctuations are driven by a variable temperature at the boundary of the layer. To start with, let us recall the analytical solution of the following problem: determine the time-evolving temperature field $T(y,t)$ in a semi-infinite solid (here: $y>0$) with the initial condition $T(y,t=0)=0$, driven by the time-periodic changes of temperature (with a magnitude Θ_0 and frequency ω) at its surface:

$$T(y=0,t)=\Re[\Theta_0\mathrm{e}^{-\mathrm{i}\omega t}]=\Theta_0\cos\omega t. \tag{6}$$

(NB: although only real parts of relevant quantities have a physical meaning, the use of complex variables simplifies both the notation and the solution procedure.) The task is in fact formally identical to the classical problem of a semi-infinite viscous fluid adjacent to an oscillating wall. We assume that the oscillation period $2\pi/\omega$ is long enough so as not to defy the Fourier hypothesis on heat transfer. The solution of (2) is readily found; it is written for the fluctuating temperature since the mean is zero everywhere:

$$\theta(y,t)=\Re\left[\Theta_0\mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{-\mathrm{i}\omega t}\right]=\Theta_0\mathrm{e}^{-y/\delta}\cos\left(\frac{y}{\delta}-\omega t\right) \tag{7}$$

where $\delta=(2\alpha_2/\omega)^{1/2}$ is called the penetration depth and $k=(1+\mathrm{i})/\delta$ is the corresponding (complex) wave number.

Two other related problems of interest consist in temperature propagation in a finite layer of solid of thickness d. Now, the b.c. on the surface $y=0$ are the same as before, Eq. (6), and the b.c. on the backside $y=d$ are stated as either isothermal or adiabatic:

$$T(y=d,t)=0\quad\text{or}\quad(\partial T/\partial y)(y=d,t)=0. \tag{8}$$

As the mean temperature in a solid is constant, the respective solutions for the fluctuating temperature are

$$\theta(y,t)=\Re\left[\Theta_0\frac{\sin k(d-y)}{\sin kd}\mathrm{e}^{-\mathrm{i}\omega t}\right]\quad\text{and}$$
$$\theta(y,t)=\Re\left[\Theta_0\frac{\cos k(d-y)}{\cos kd}\mathrm{e}^{-\mathrm{i}\omega t}\right]. \tag{9}$$

Let us focus first on detailed analytical expressions for the semi-infinite case. The time averaging of θ^2 over the oscillation period $2\pi/\omega$ yields the variance of thermal fluctuations. It is computed from Eq. (7):

$$\langle\theta^2(y,t)\rangle=\frac{1}{2}\Theta_0^2\mathrm{e}^{-2y/\delta}. \tag{10}$$

Analogously, the destruction rate of $\langle\theta^2\rangle/2$, cf. Eq. (5), is computed as

$$\alpha_2\left\langle\frac{\partial\theta}{\partial x_j}\frac{\partial\theta}{\partial x_j}\right\rangle=\Theta_0^2\mathrm{e}^{-2y/\delta}\frac{\alpha_2}{\delta^2}=\frac{\langle\theta^2\rangle}{\delta^2/2\alpha_2}. \tag{11}$$

It is readily verified from Eqs. (10) and (11) that the balance of Eq. (5) is identically satisfied. Moreover, as it transpires from (11), the destruction rate of thermal fluctuations can be expressed in the form $\langle\theta^2\rangle/\tau_\theta$, with the characteristic time scale being $\tau_\theta=\tau_{\theta,\infty}$ where $\tau_{\theta,\infty}=1/\omega$ (the subscript ∞ denotes the value obtained for the semi-infinite layer of solid). This observation may seem trivial since in the problem considered ω is the only parameter that can determine the time scale. However, in analogy to scalar mixing issue in PDF methods (Pope, 2000), the form of the destruction term directly suggests a possible model for particle simulation, i.e., a relaxation of the instantaneous temperature towards the mean (here: zero) with a time scale τ_θ, cf. Eq. (14) below.

3.2. Monte Carlo simulation

In the Lagrangian particle approach, the motion of discrete portions of internal energy (particles) is tracked, cf. Pozorski et al. (2004) for a detailed explanation. Particles are ascribed the instantaneous temperature T and location $\mathbf{X}$; formally, the one-point scalar PDF $f=f_{T\mathbf{X}}(\Theta,\mathbf{x},t)$ can be introduced; here, Θ and $\mathbf{x}$ are respective sampling space variables and $f(\Theta,\mathbf{x},t)\,\mathrm{d}\Theta$ is the probability of the event that in a given flow realisation the temperature T at the point $(\mathbf{x},t)$ is contained in the range $\Theta\leqslant T<\Theta+\mathrm{d}\Theta$. The governing equation for f is derived from Eq. (2), cf. Pope (2000), and further rearranged as follows:

$$\frac{\partial f}{\partial t}=\alpha_2\frac{\partial^2 f}{\partial x_k\partial x_k}-\frac{\partial^2}{\partial\Theta^2}\left[\left\langle\alpha_2\frac{\partial T}{\partial x_k}\frac{\partial T}{\partial x_k}\middle|T=\Theta\right\rangle f\right]; \tag{12}$$

here, the symbol $\langle\cdot|\cdot\rangle$ stands for the conditional average. Regarding the physical meaning, the first RHS term of Eq. (12) describes the diffusion in space and the second RHS term represents the anti-diffusion in the phase space of Θ. The latter corresponds to the fact that in the absence of production term within a solid (the thermal fluctuations are due to forcing at the solid–fluid interface) the temperature fluctuations will decay and the instantaneous temperature will relax its local mean value. Concerning now the modelling, the heat equation in a solid is solved by the random walk of particles, corresponding to the first RHS term of Eq. (12), and the particle temperatures change to account for the second RHS term of Eq. (12), here approximated by a simple relaxation model (cf. Pope, 2000 or Jones, 2002). In a statistically 1D setting, the locations and temperatures of stochastic particles evolve as

$$\mathrm{d}Y = \sqrt{2\alpha_2}\,\mathrm{d}W, \tag{13}$$

$$\mathrm{d}T = -\frac{T-\langle T\rangle}{\tau_\theta}\,\mathrm{d}t, \tag{14}$$

where $\mathrm{d}W$ is an increment of the Wiener process and τ_θ is a scalar mixing time scale (to be determined). For a thorough (yet practical) perspective on stochastic processes, the reader may wish to refer to Gardiner (1990). Here, let us only state that in the numerical implementation of the approach equations (13) and (14) are solved for a discrete time step, Δt; the finite increment of the Wiener process is simulated as $\Delta W = \xi_{\mathscr{N}}\sqrt{\Delta t}$ with $\xi_{\mathscr{N}}$ being a random number from the standard normal (Gaussian) distribution. The system of particle equations corresponds to a closed evolution equation for the PDF. The mean heat equation and a balance of the temperature variance (also higher-order moments, if needed) are retrieved from it upon suitable integration.

The Monte Carlo computation of the temperature inside the wall material is performed for the finite layer d of solid. The inlet boundary conditions at the interface $y=0$ are formulated for particles that leave the computational domain on this side and are reflected back. The b.c. are meant to yield the correct mean temperature and the fluctuating temperature variance (equal to $\Theta_0^2/2$) at the interface; this can be achieved by setting the temperature of incoming particles to

$$T_{\mathrm{in}} = \frac{\Theta_0}{\sqrt{2}}\,\xi_{\mathscr{N}} \quad \text{or} \quad T_{\mathrm{in}} = \Theta_0\cos(2\pi\xi_{\mathscr{U}}). \tag{15}$$

The first expression ensures that the temperature PDF of incoming particles is Gaussian ($\xi_{\mathscr{N}}$ is again a random number from the standard normal distribution). As an alternative, the second expression uses the forcing defined from Eq. (6) with a random phase ($\xi_{\mathscr{U}}$ is a random number from the standard uniform distribution). We have tested in Monte Carlo computations that the resulting profile of $\langle\theta^2\rangle(y)$ is not affected by either choice of b.c. from formula (15). On the other boundary ($y=d$) two variants of boundary conditions, isothermal and adiabatic, have been imposed. In particle variables, this is respectively done as

$$T_{\mathrm{in}} = 0 \quad \text{or} \quad T_{\mathrm{in}} = T_{\mathrm{out}}. \tag{16}$$

For both boundaries, $y=0$ and $y=d$, the reflected Brownian motion has been accounted for by applying the b.c. to the particles that left the domain at a given time step but also (with a non-zero probability) to some of those that did not, i.e., the particles that left out and returned during the time step (cf. Wacławczyk et al., 2004).

To get a better insight, the simulations have been performed for three different cases: $d \gg \delta$, $d \sim \delta$, and $d \ll \delta$. Preliminary results (not shown here) for the decay of fluctuating temperature variance with depth present a growing discrepancy with respect to analytical predictions with the decreasing ratio of d/δ. They have been obtained with a constant time scale $\tau_{\theta,\infty}$ in the scalar mixing model (otherwise correct in the case $d=\infty$). A remedy proposed here consists in taking a variable time scale, $\tau_\theta = \tau_\theta(y)$, determined as follows. The temperature relaxation term in the evolution of stochastic particles, Eq. (14), produces a modelled destruction term $-2\langle\theta^2\rangle/\tau_\theta$ in the transport equation for the temperature variance. Comparison with the exact equation for $\langle\theta^2\rangle$, Eq. (5), leads to the prescription of the relaxation time:

$$\tau_\theta = \langle\theta^2\rangle\left[\alpha_2\left\langle\frac{\partial\theta}{\partial x_j}\frac{\partial\theta}{\partial x_j}\right\rangle\right]^{-1}. \tag{17}$$

For the isothermal and adiabatic b.c. on the back surface ($y=d$) the respective formulae are found from Eq. (17) with the use of Eq. (9):

$$\tau_\theta = \frac{1}{\alpha_2}\left|\frac{\sin k(d-y)}{\sin kd}\right|^2\cdot\left|\frac{k\cos k(d-y)}{\sin kd}\right|^{-2} \quad \text{and}$$

$$\tau_\theta = \frac{1}{\alpha_2}\left|\frac{\cos k(d-y)}{\cos kd}\right|^2\cdot\left|\frac{k\sin k(d-y)}{\cos kd}\right|^{-2}. \tag{18}$$

The resulting profiles of the time scale $\tau_\theta(y)$, normalised by the forcing frequency ω, are shown in Fig. 1 for the isother-

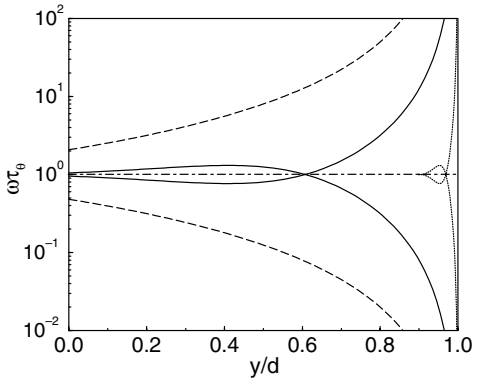

Fig. 1. Analytical profiles of the position-dependent time scale τ_θ (normalised with the forcing frequency ω). Various ratios of the layer thickness d to the penetration depth δ: $d/\delta = 0.5$ (dashed lines), $d/\delta = 2$ (solid lines), $d/\delta = 25$ (dotted lines), $d/\delta = \infty$ (horizontal dash-dotted line). Ascending lines: upper boundary ($y=d$) adiabatic, descending lines: upper boundary isothermal.

mal and adiabatic condition on the back surface and for three different ratios of the layer thickness d to the penetration depth δ. It is readily noticed that τ_θ considerably departs from $1/\omega$, specially for small ratios of d/δ. At the backside wall (i.e., in the limit $y \to d$) the time scale tends asymptotically either to zero (isothermal wall, rapid attenuation of any fluctuations there) or to infinity (adiabatic wall, non-zero fluctuations can persist there).

The prescription of Eq. (17) makes it now possible to also close the Eulerian equation for the temperature fluctuations in solid, Eq. (5), as

$$\frac{\partial\langle\theta^2\rangle}{\partial t} = \alpha_2 \frac{\partial^2\langle\theta^2\rangle}{\partial x_j \partial x_j} - \frac{2}{\tau_\theta}\langle\theta^2\rangle, \tag{19}$$

where the time scale from Eq. (18) has the form $\tau_\theta = \tau_\theta(\omega, y/d, d/\delta)$, shown in Fig. 1 and dependent also on the thermal b.c. type. Basically, Eq. (19) can be used in RANS, specially for routine computations (when the temperature PDF is not needed) that certainly are less CPU-intensive than the Monte Carlo method used here. Yet, the problem of stating appropriate b.c. at the backside wall ($y = d$) will seemingly persist in RANS for cases other than the isothermal wall with zero temperature fluctuations.

The PDF computation results of the model (13) and (14) with $\tau_\theta(y)$, Eq. (18), are presented in Fig. 2. The theoretical predictions of $\langle\theta^2\rangle$ resulting from Eqs. (9) are quite well matched. We emphasise that the spatially variable relaxation time scale for temperature in solid material is meant to ensure an approximately correct attenuation of thermal fluctuations that are due to external forcing; arguably, it is a necessary ingredient of a local (one-point) model. Otherwise, the position-dependent relaxation rate in a homogeneous material has no sound physical explanation in terms of underlying small-scale phenomena in a solid. Yet, our main aim here has been to construct a model that deals with the solid part of a coupled fluid–solid system. The model will be given a posteriori support through the conjugate heat transfer computations.

4. Stochastic particle approach for conjugate heat transfer

4.1. Remarks on particle methods for heat transfer

Before we describe the actual computational method chosen, let us briefly mention some other particle approaches of potential interest for the conjugate heat transfer, worth further exploration. Basically, in particle methods there are two ways to simulate the diffusion process in physical space: either a fixed amount of a physical property (vorticity, temperature, mass, etc.) is associated with stochastic particles and this property diffuses through a random walk of particles (possibly superposed with advection), or the particle property is allowed to change through a non-local interaction with other particles. The former idea (that of a random walk) has been proposed in 2D vortex methods in fluid mechanics. The random walk is also an ingredient of our model; yet, due to the interaction with mean fields the particle properties (in particular: temperature) are bound to change. The latter idea (that of interactions between particles) has been proposed in vortex methods and in smoothed particle hydrodynamics (SPH). In some variants of vortex methods for viscous flows, the technique of resampling or particle strength exchange is used (cf. Cottet and Koumoutsakos, 2000). In the formulation of SPH (Monaghan, 1992) the idea of interactions between particles has been further developed for multi-material heat transfer (Cleary, 1998). In that approach, particles representing the solid material remain immobile, but their temperature changes through a modelled heat diffusion (Laplacian) term. There is no difficulty in satisfying the conditions on the material interfaces.

In random walk methods for complex diffusion problems (Ghoniem and Sherman, 1985) two kinds of particles are sometimes used; each group is carrying a separate variable. Similarly, vortex method computations with thermal field reported in the literature (they are quite rare, cf. Ogami, 2001) use two classes of particles, separately for vorticity and temperature. Yet, apart from being more

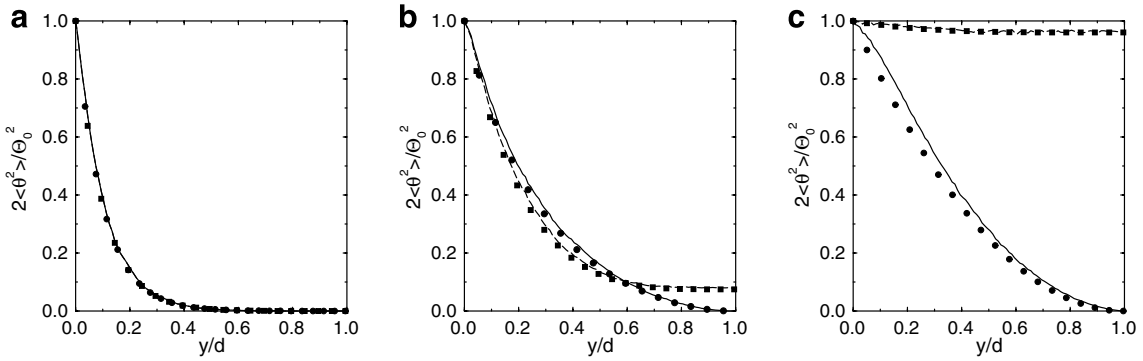

Fig. 2. Fluctuating temperature variance in the solid layer with forced lower boundary ($y = 0$). Computation with the position-dependent time scale τ_θ and various ratios of the layer thickness to the penetration depth: (a) $d/\delta = 5$, (b) $d/\delta = 2$, (c) $d/\delta = 0.5$. Simulation results with boundary ($y = d$) isothermal (solid lines) or adiabatic (dashed lines). Symbols (●, ■): respective theoretical predictions.

computationally expensive, such an approach seems impractical for turbulent heat transfer computations, since no joint moments (like $\langle u_i\theta\rangle$) are available.

In the standard random walk of "temperature elements", particles evolve according to

$$\mathrm{d}\mathbf{X} = \sqrt{2\alpha}\,\mathrm{d}\mathbf{W}, \tag{20}$$

$$\mathrm{d}T = 0. \tag{21}$$

Averaging over particles yields the mean temperature field. This approach is particularly useful for problems of diffusion from point or line sources where Eulerian methods face difficulties due to resolution problems in regions of steep gradients. On the other hand, in a gradient random walk each particle n evolving by Eq. (20) will carry a temperature jump δT_n; cf. Ghoniem and Sherman (1985) for details. The advantage of the gradient walk approaches is that the field of interest (e.g., temperature or velocity) is respectively obtained through summation of the temperature jump contributions associated with particles or through the computation of induced velocity field (vortex elements), finally giving a smoothed field.

4.2. Main idea of the present approach

A concept proposed here is to solve the conjugate heat transfer problem in a coupled Monte Carlo approach. Stochastic particles carry a certain amount of internal energy and represent both the heat transfer (convective and conductive) in fluid and the heat conduction in solid. This point is illustrated in Fig. 3. The building blocks of the approach are: the scalar PDF method for near-wall turbulence (Pozorski et al., 2004), a stochastic method for unsteady heat conduction in solid (cf. previous section), and the statement of appropriate thermal consistency conditions for particle temperature at the wall (at the solid–fluid interface).

Concerning the estimation of thermal fluctuations in solid, thus far we do not see any constructive possibility to use either the simple random walk, Eqs. (20) and (21), or the gradient random walk of Section 4.1 for the purpose. Rather, we stick to the model proposed in Section 3. Two questions need to be addressed before the conjugate heat transfer computation is attempted with that "building block" model for solid material and another building block model, that of the scalar PDF, for fluid (Pozorski et al., 2004). First question relates to the choice of the forcing frequency, and the second question pertains to the implementation of the interface condition. Both points are addressed now.

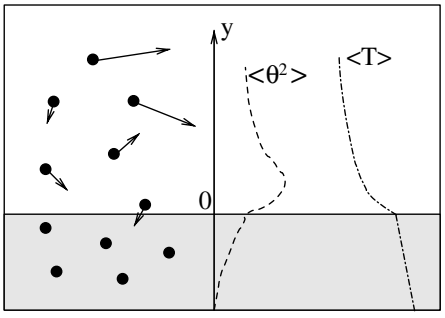

Fig. 3. Schematic picture: stochastic particle method for the coupled heat transfer at the solid–fluid interface (shaded area: solid material).

4.3. Forcing frequency for unsteady heat conduction

The model for thermal fluctuations in the solid, introduced in Section 3, needs a forcing frequency as an external parameter. It characterises near-wall vortical structures in turbulent flow. The DNS of Tiselj et al. (2001b) for $Pr = 7$ provides some indications as to the choice of ω. For given thermal activity ratio $K = 1$, and the thickness of solid layer $d^{++} = 20$, we have compared the profile of the thermal fluctuation intensity $\theta_{\mathrm{rms}} = \langle\theta^2\rangle^{1/2}$ in the solid, as given by the DNS, with the prediction resulting from Eq. (9) where δ is uniquely determined by ω and α_2; in wall units:

$$(\delta^{++})^2 = \frac{2}{Pr\,\omega^+}. \tag{22}$$

(NB: the non-dimensional length in solid is introduced as $y^{++} = y^+\sqrt{\alpha_1/\alpha_2}$.) The comparison gives the correct level of $\langle\theta^2\rangle$ at the backside wall for $d/\delta \approx 3.2$–3.8. Hence the characteristic time scale is $\tau^+ = 1/\omega^+ \approx 100$–$140$. Finally, we have adopted the time scale (external input data for the stochastic model) of $\tau^+ = 120$. The corresponding penetration depth results from Eq. (22): $\delta^{++} = 5.75$, keeping in mind that this particular value is charged with uncertainty of about 10%.

4.4. Interface conditions for temperature

Conditions for the instantaneous temperature at the fluid–solid interface are a consequence of the physical picture of the conjugate heat transfer. In the statistical approach, the boundary (interface) conditions are formulated for the averaged variables. It transpires from Eq. (3) that the mean temperature and the mean heat flux are continuous at the interface $y = 0$. Following the identity $\langle\theta^2\rangle = \langle T^2\rangle - \langle T\rangle^2$, the temperature variance is also continuous at the interface:

$$\langle\theta^2\rangle|_{y=0^-} = \langle\theta^2\rangle|_{y=0^+}. \tag{23}$$

Concerning the condition at the solid–fluid interface in the particle setting, at first we considered three different possibilities: (a) the particles are allowed to cross the interface either way (i.e., penetrate into the other medium) with possibly a temperature change, (b) the particles cross the interface with property/weight change, and (c) the particles are reflected from the interface (no penetration) with some communication (to be devised) with the other medium. Upon further reflexion, the option (a) has been implemented as detailed below.

Consider first a particle approach to the heat equation in a finite layer $0 \leqslant y \leqslant d$ of solid with isothermal walls of

temperatures T' and T'' and the thermal diffusivity α. For a homogeneous material the heat flux in this simple case is obviously $\lambda(T'' - T')/d$. Suppose now that the thermal conductivity changes in a smooth manner within the layer, $\lambda = \lambda(y)$. Then the constant heat flux $q = \lambda(y)\,\mathrm{d}T/\mathrm{d}y = \text{const}$ at any section y implies that $\mathrm{d}q = 0$ and, consequently,

$$\frac{\mathrm{d}^2T}{\mathrm{d}y^2} = -\frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}y}\frac{\mathrm{d}T}{\mathrm{d}y}. \tag{24}$$

The particle $\mathrm{d}T$ equation (cf. below) with the mixing term (here to be applied with the short time scale $\tau_\theta \to 0$ to result in zero fluctuations) and the isothermal b.c. described before yield the linear temperature distribution in the layer, unless the variability of λ is somehow accounted for. We propose a following modification that can be justified by the Taylor series expansion of the temperature profile around y_0 (that can be thought of as the current location of a stochastic particle) in the neighbourhood of $\mathrm{d}y$ (that can be thought of as a distance covered by the particle at a single time step):

$$T(y_0 + \mathrm{d}y) = T(y_0) + \frac{\mathrm{d}T}{\mathrm{d}y}\,\mathrm{d}y + \frac{1}{2}\frac{\mathrm{d}^2T}{\mathrm{d}y^2}(\mathrm{d}y)^2. \tag{25}$$

We now argue that the linear term is already simulated by the standard $\mathrm{d}T$ equation, so an additional temperature increment is necessary for a possible departure of temperature from the linear profile in the solid layer:

$$\mathrm{d}T_\lambda = \frac{1}{2}\frac{\mathrm{d}^2T}{\mathrm{d}y^2}(\mathrm{d}y)^2 = -\frac{1}{2}\frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}y}\,\mathrm{d}y\frac{\mathrm{d}T}{\mathrm{d}y}\,\mathrm{d}y; \tag{26}$$

the second equality follows by virtue of Eq. (24).

In the conjugate heat transfer problems, the case of interest is in particular the discontinuity (jump) of λ at the solid–fluid interface. Suppose that a particle crosses the interface at certain time step. Then, to the standard $\mathrm{d}T$ equation the particle temperature change should be added, based on Eq. (26) with the term $1/\lambda$ replaced by its average value at the interface, the term $(\mathrm{d}\lambda/\mathrm{d}y)\,\mathrm{d}y$ replaced by the jump $(\lambda_1 - \lambda_2)$, and by taking the absolute value of $\mathrm{d}y$ so that the formula remains unchanged for either direction of the interface crossing (from the region of smaller to larger λ or the opposite):

$$\mathrm{d}T_\lambda = -\frac{1}{4}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)(\lambda_1 - \lambda_2)\frac{\mathrm{d}T}{\mathrm{d}y}|\mathrm{d}y|. \tag{27}$$

The system of particle equations for heat conduction in solid is

$$\mathrm{d}Y = \sqrt{2\alpha_2}\,\mathrm{d}W, \tag{28}$$

$$\mathrm{d}T = -\frac{T - \langle T\rangle}{\tau_\theta}\,\mathrm{d}t + \mathrm{d}T_\lambda. \tag{29}$$

To validate the proposal for variable λ, we have performed two test computations in a solid layer $0 \leqslant y \leqslant 1$ with the b.c. $T(0) = 0$ and $T(1) = 1$. In the first test case, the following profile of thermal conductivity is assumed: $\lambda(y) = y + 1$. It results in a logarithmic temperature distribution in the layer: $T(y) = C\log(y + 1)$ where $C = 1/\log(2)$. In the second computation, a jump from $\lambda_1 = 1$ to $\lambda_2 = 2$ is assumed at $y = 0.5$, resulting in a broken line profile of the temperature: $T(y) = 4y/3$ for $y \leqslant 0.5$ and $T(y) = 2(y + 0.5)/3$ otherwise. As illustrated in Fig. 4, the computation results agree well with the theoretical predictions for both test cases.

As far as conjugate heat transfer computations are concerned, the above result provides us with a support to include there the suggested additional term accounting for varying thermal conductivity. Namely, for stochastic particles crossing the fluid–solid interface (from either side), the temperature jump is added as described by Eq. (27) with the mean temperature gradient used instead of $\mathrm{d}T/\mathrm{d}y$.

4.5. Complete model of conjugate heat transfer

To determine the temperature field at the fluid side the scalar PDF approach (cf. Pope, 2000; Jones, 2002) is applied with further developments for the near-wall region

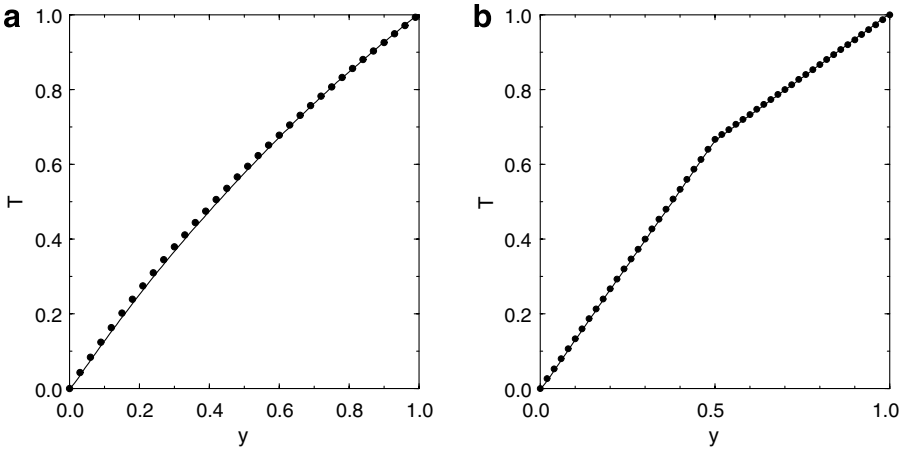

Fig. 4. Test case: particle computation of heat conduction in a finite solid layer of variable heat conductivity: (a) a continuous profile of λ; (b) a finite jump in λ. Analytical profiles of temperature: symbols, computation results: lines.

as detailed in Pozorski et al. (2004). The motion and temperature evolution of stochastic particles in fluid are governed by equations:

$$\mathrm{d}X_i = \langle U_i \rangle \,\mathrm{d}t + \frac{\partial \alpha_t}{\partial x_i}\,\mathrm{d}t + \sqrt{2(\alpha_1 + \alpha_t)}\,\mathrm{d}W_i, \tag{30}$$

$$\mathrm{d}T = -\left(\frac{1}{\tau_\phi} + \frac{1}{\tau_M}\right)(T - \langle T \rangle)\,\mathrm{d}t + \mathrm{d}T_\lambda, \tag{31}$$

where τ_ϕ is the scalar mixing time scale, τ_M accounts for the near-wall molecular transport effects, and α_t is the turbulent heat diffusivity. In the scalar PDF approach, $\alpha_t = \nu_t/Pr_t$ is determined from the turbulent viscosity and the profile of the turbulent Prandtl number that depends on the type of the wall boundary condition, cf. Pozorski et al. (2004). For the computations reported in the paper, we used the necessary velocity statistics from the DNS data of Moser et al. (1999). Otherwise, no external input is needed in the standalone, joint velocity-scalar PDF approach to turbulent flows with heat transfer (Pozorski et al., 2003a,b, 2004).

Stochastic particles in solid evolve according to

$$\mathrm{d}X_i = \sqrt{2\alpha_2}\,\mathrm{d}W_i, \tag{32}$$

$$\mathrm{d}T = -\frac{T - \langle T \rangle}{\tau_\theta}\,\mathrm{d}t + \mathrm{d}T_\lambda \tag{33}$$

where τ_θ is determined from Eq. (18b) for the case of the adiabatic backside wall considered here.

In both Eqs. (31) and (33) the term $\mathrm{d}T_\lambda$, known from Eq. (27), in homogeneous media (constant λ_1 and λ_2) matters only for particles that cross the interface at a given time step. In particular, for the 1D conjugate case with a flat solid–fluid interface, the jump of particle temperature at the interface is determined as

$$\mathrm{d}T_\lambda = -\frac{1}{4}\left(K - \frac{1}{K}\right)\frac{\mathrm{d}\langle T \rangle}{\mathrm{d}\tilde{y}}\,\mathrm{d}\tilde{y}, \tag{34}$$

where $\tilde{y}$ is the scaled wall-normal coordinate: $\tilde{y} = y\sqrt{\alpha_1/\alpha_2}$ in solid $y < 0$, while $\tilde{y} = y$ in fluid $y > 0$. The boundary condition at the lower part of the solid, $y = -d$ say (the bottom horizontal line in Fig. 3), is implemented here as the constant flux condition.

5. Computation results for heated channel flow

The near-wall region of a fully-developed, heated channel flow at $Re_\tau = 180$ (based on the friction velocity and the channel half-width) bounded by solid walls of a finite thickness is considered as the validation case. The computational domain extends from the channel centerline ($y^+ = 180$) through the fluid–solid interface ($y = 0$), down to the outer wall at $y^{++} = -20$ (the lowest horizontal line in Fig. 3). Scalar PDF computations of conjugate heat transfer in near-wall turbulence have been performed for several choices of relevant physical parameters; the second-order thermal statistics (the fluctuating temperature variance) are of particular interest.

Typically, the computations have been performed with about 10^4 stochastic particles and a standard run (on a present-day, single-processor personal computer) required a few tens of minutes of the CPU time, depending on the choice of numerical parameters. The sensitivity of results to the molecular Prandtl number, the thermal boundary condition on the interface (determined by the thermal activity ratio of fluid and solid), and the solid wall thickness has been studied. Some qualitative results are reported first. Fig. 5(a) shows computed non-dimensional mean temperature profiles in solid and fluid for some values of the thermal activity ratio K. Obviously, the temperature varies linearly in solid, and the behaviour in fluid is alike that previously computed for limiting (isothermal and isoflux) cases. This fact is well supported by the DNS. Next, Fig. 5(b) presents a profile of the fluctuating temperature r.m.s. (normalised by the wall level) for $K = 10$. In turbu-

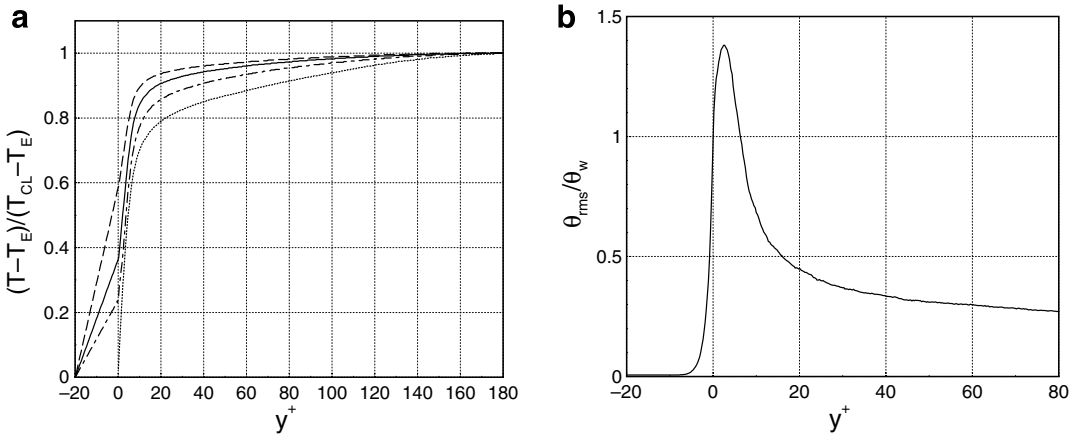

Fig. 5. Heated turbulent channel flow at $Re_\tau = 180$ bounded by finite thickness ($d^{++} = 20$) solid walls. (a) Mean temperature profile: isothermal wall (dotted line), finite thermal activity ratio: $K = 1$ (dashed), $K = 5$ (solid), $K = 10$ (dash-dotted); (b) fluctuating temperature r.m.s. in fluid and solid ($K = 10$). Results made nondimensional with the outer wall temperature T_E, channel centerline temperature T_CL, and the r.m.s. fluctuating temperature at the wall θ_w, respectively.

lent flow, the profile is similar to that computed previously for fluid-only case. On the other hand, the temperature r.m.s. in solid decays roughly exponentially with depth and then levels off (because of the isoflux thermal condition at the outer wall).

As far as quantitative comparison is concerned, computational results of our conjugate heat transfer model are validated against the data of Tiselj et al. (2001b) from the DNS in fluid coupled with unsteady heat conduction in solid. The comparison indicates that there is a fair agreement between the two; yet, some discrepancies persist. The r.m.s. temperature fluctuations in fluid and in solid are shown in Fig. 6 for $Pr = 7$, $K = 1$, and two wall depths. On the fluid side, the maximum in the buffer layer is shifted towards the wall. In the solid, the decay of thermal fluctuations with depth is too fast as compared to the DNS; yet, the solid-only stochastic model shown previously has produced very good agreement. The reason for the discrepancy may be twofold. Contrary to the previous case, there is now a mean temperature gradient in the wall material that may have some influence. Moreover, all particles that penetrate the solid from the fluid side enter with their own instantaneous temperature; in the previous stochastic model for the solid only, the temperature of those particles was prescribed from a certain distribution law to preserve correct r.m.s. fluctuating temperature level at the wall. Arguably, both these issues need a further insight.

Results of the computations with given wall depth ($d^{++} = 20$) and varying K are presented in Fig. 7. Here again, there is a qualitative agreement when compared to the reference data from the DNS. Yet, one of the numerical difficulties that we have encountered relates to formula (34); there, for the values of K much smaller or much larger than unity, the temperature jump $\mathrm{d}T_\lambda$ at the interface may

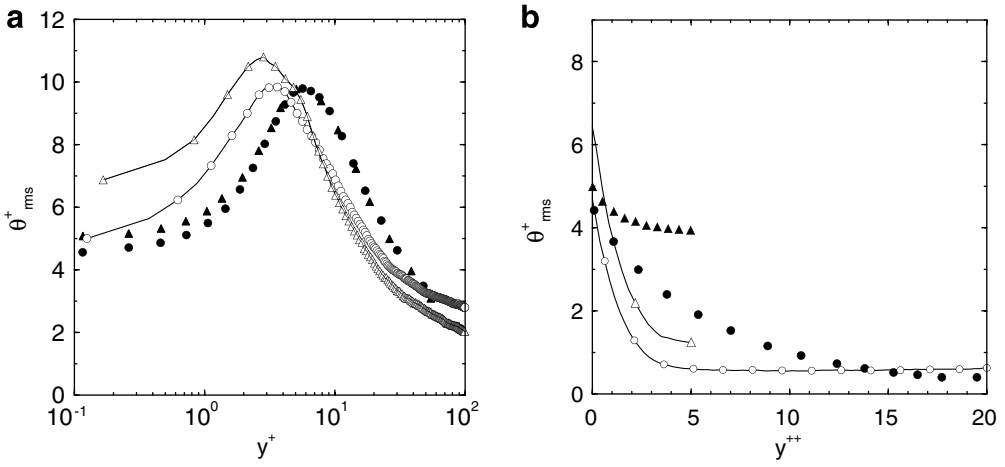

Fig. 6. Heated turbulent channel flow at $Re_\tau = 150$, $Pr = 7$, and the thermal activity ratio $K = 1$: The r.m.s. temperature fluctuations: (a) in fluid, (b) in solid. Symbols: DNS data (Tiselj et al., 2001b) for the wall thickness $d^{++} = 5$ (▲) and $d^{++} = 20$ (●). Lines with corresponding open symbols: PDF computation.

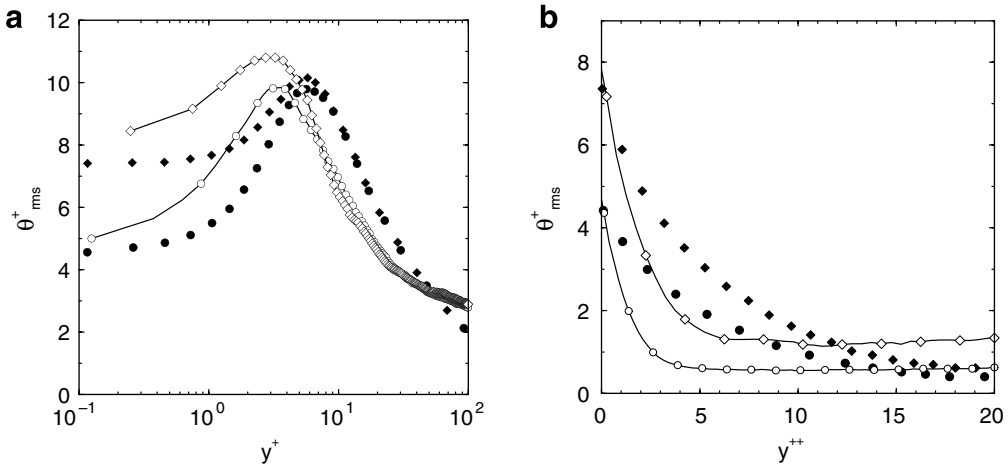

Fig. 7. Heated turbulent channel flow at $Re_\tau = 150$, $Pr = 7$, and the wall thickness $d^{++} = 20$. The r.m.s. temperature fluctuations: (a) in fluid, (b) in solid. Symbols: DNS data (Tiselj et al., 2001b) for the thermal activity ratio $K = 5$ (♦), $K = 1$ (●). Lines with corresponding open symbols: PDF computation.

become relatively large when compared to the temperature scale (taken as the r.m.s. fluctuating wall temperature), hence calling for very small time steps.

6. Conclusion

New modelling proposals for turbulent flows with the temperature field have been advanced in the paper. A stochastic particle model has been proposed for thermal fluctuations in solid material; they are generated by a forcing on its surface. As a spin-off of the analysis, a modelled equation for thermal fluctuation variance in solid has been obtained; it can be solved, together with a statistical turbulence model for fluid, in the Eulerian approach to conjugate heat transfer, alleviating thus the need of prescribing the level of temperature variance at the solid–fluid interface. Yet, since the objectives of the present contribution are different, such a RANS computation has not been attempted here. Instead, the stochastic approach has been followed both in modelling and for numerical computations. The results for a solid layer compare favourably with analytical predictions of the decay of fluctuating temperature variance with depth. The model represents a first building block of a general stochastic approach to the conjugate heat transfer. In the one-point scalar PDF method proposed here, the external input parameter to the model, i.e., the forcing frequency (or the appropriate near-wall time scale of thermal fluctuations) has to be estimated. The DNS data of conjugate heat transfer (Tiselj et al., 2001b), in particular the decay of the r.m.s. fluctuating temperature in solid material, provide a convincing support for the time scale and to the stochastic modelling with a single forcing frequency that is also a dominant frequency in temperature spectrum at the wall. Interface conditions (i.e., consistency conditions for temperature statistics on the fluid–solid dividing surface) have been formulated in particle setting. In particular, the variation of heat conductivity coefficient (either continuously changing in the solid material or with a finite jump at the interface) has been accounted for. The complete model for fluid and solid yields reasonable results (compared to recent DNS) for the mean temperature and the fluctuating temperature variance in a coupled case for the heated channel flow with finite-thickness walls.

Inevitably, for the temperature considered as a passive scalar the temperature frequency spectrum at the wall heavily depends on that of the wall-normal velocity. Any stochastic model able to provide an estimation of the temperature spectrum must contain some information on turbulence structure and thus goes beyond the one-point closure exploited here. The structural turbulence model in unsteady 2D formulation (Kasagi et al., 1989) may be a departure point for further studies in that direction. In a similar vein, a stochastic approach with large-scale or the scalar filtered density function (FDF) model combined with the dynamics of the large-eddy velocity modes in the near-wall region seems promising. Such a formulation with large-scale structures computed from the proper orthogonal decomposition (POD) of the velocity field has recently been proposed as a hybrid FDF/POD model (Wacławczyk and Pozorski, 2004). It provides insight into instantaneous structures of flow and thermal fields at varying Prandtl numbers and can hopefully be refined for conjugate heat transfer phenomena. Analogous developments for velocity dynamics computed from LES, i.e., a hybrid, near-wall FDF/LES formulation are in progress. To take a full advantage of the joint PDF approach, such as exact representation of convective and chemical source terms, further developments include hybrid (zonal) models and the modelling of heterogeneous chemical reactors in turbulent flow (e.g., surface reactions of catalytic type).

References

Cleary, P.W., 1998. Modelling confined multi-material heat and mass flows using SPH. Appl. Math. Modell. 22, 981–993.

Cottet, G.H., Koumoutsakos, P., 2000. Vortex Methods, Theory and Practice. Cambridge University Press.

Gardiner, C.W., 1990. Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences, corrected second ed. Springer-Verlag, Berlin.

Ghoniem, A.F., Sherman, F.S., 1985. Grid-free simulation of diffusion using random walk methods. J. Comput. Phys. 61, 1–37.

Jones, W.P., 2002. The joint scalar probability density function. In: Launder, B.E., Sandham, N.D. (Eds.), Closure Strategies for Turbulent and Transitional Flows. Cambridge University Press, pp. 582–625.

Kasagi, N., Kuroda, A., Hirata, M., 1989. Numerical investigation of near-wall turbulent heat transfer taking into account the unsteady heat conduction in the solid wall. J. Heat Transfer 111, 385–392.

Kasagi, N., Tomita, Y., Kuroda, A., 1992. Direct numerical simulation of passive scalar field in a turbulent channel flow. J. Heat Transfer 114, 598–606.

Kawamura, H., Abe, H., Matsuo, Y., 1999. DNS of turbulent heat transfer in channel flow with respect to Reynolds and Prandtl number effects. Int. J. Heat Fluid Flow 20, 196–207.

Kawamura, H., Ohsaka, K., Abe, H., Yamamoto, K., 1998. DNS of turbulent heat transfer in channel flow with low to medium-high Prandtl number fluid. Int. J. Heat Fluid Flow 19, 482–491.

Kong, H., Choi, H., Lee, J.S., 2000. Direct numerical simulation of turbulent thermal boundary layers. Phys. Fluids 12, 2555–2568.

Launder, B.E., 1996. An introduction to single-point closure methodology. In: Gatski, T.B., Hussaini, M.Y., Lumley, J.L. (Eds.), Simulation and Modeling of Turbulent Flows. Oxford University Press, pp. 243–310.

Lu, D.M., Hetsroni, G., 1995. Direct numerical simulation of a turbulent open channel flow with passive heat transfer. Int. J. Heat Mass Transfer 38, 3241–3251.

Monaghan, J.J., 1992. Smoothed particle hydrodynamics. Annu. Rev. Astron. Astrophys. 30, 543–574.

Moser, R.D., Kim, J., Mansour, N.N., 1999. Direct numerical simulation of turbulent channel flow up to $Re_\tau = 590$. Phys. Fluids 11, 943–945.

Mosyak, A., Pogrebnyak, E., Hetsroni, G., 2001. Effect of constant heat flux boundary condition on wall temperature fluctuation. J. Heat Transfer 123, 213–218.

Na, Y., Papavassiliou, D.V., Hanratty, T.J., 1999. Use of direct numerical simulation to study the effect of Prandtl number on temperature fields. Int. J. Heat Fluid Flow 20, 187–195.

Nagano, Y., 2002. Modelling heat transfer in near-wall flows. In: Launder, B.E., Sandham, N.D. (Eds.), Closure Strategies for Turbulent and Transitional Flows. Cambridge University Press, pp. 188–247.

Ogami, Y., 2001. Simulation of heat-fluid motion by the vortex method. JSME Int. J. B – Fluid 44, 513–519.

Polyakov, A.F., 1974. Wall effect on temperature fluctuations in the viscous sublayer (in Russian). Teplofizika Vysokikh Temperatur 12, 328–337.

Pope, S.B., 2000. Turbulent Flows. Cambridge University Press.

Pozorski, J., Wacławczyk, M., Minier, J.P., 2003a. Full velocity–scalar probability density function computation of heated channel flow with wall function approach. Phys. Fluids 15, 1220–1232.

Pozorski, J., Wacławczyk, M., Minier, J.P., 2003b. PDF computation of heated turbulent channel flow with the bounded Langevin model. J. Turbul. 4, art. no. 011.

Pozorski, J., Wacławczyk, M., Minier, J.P., 2004. Scalar and joint velocity-scalar PDF modelling of near-wall turbulent heat transfer. Int. J. Heat Fluid Flow 25, 884–895.

Robinson, S.K., 1991. Coherent motions in the turbulent boundary layer. Annu. Rev. Fluid Mech. 23, 601–639.

Sommer, T.P., So, R.M.C., Zhang, H.S., 1994. Heat transfer modeling and the assumption of zero wall temperature fluctuations. J. Heat Transfer 116, 855–863.

Tiselj, I., Pogrebnyak, E., Li, C., Mosyak, A., Hetsroni, G., 2001a. Effect of wall boundary condition on scalar transfer in a fully developed turbulent flume. Phys. Fluids 13, 1028–1039.

Tiselj, I., Bergant, R., Mavko, B., Bajsić, I., Hetsroni, G., 2001b. DNS of turbulent heat transfer in channel flow with heat conduction in the solid wall. J. Heat Transfer 123, 849–857.

Wacławczyk, M., Pozorski, J., 2004. Conjugate heat transfer modelling using the FDF approach for near-wall scalar transport coupled with the POD method for flow dynamics. In: Andersson, H.I., Krogstadt, P.A. (Eds.), Advances in Turbulence X. CIMNE, Barcelona, pp. 253–256.

Wacławczyk, M., Pozorski, J., Minier, J.P., 2004. PDF computation of turbulent flows with a new near-wall model. Phys. Fluids 16, 1410–1422.